

PRECALCULUS MATHEMATICS IN A NUTSHELL:

**GEOMETRY,
ALGEBRA,
TRIGONOMETRY**

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CHAPTER 1
GEOMETRY

**"Any fool can know. The point is to understand."
—Albert Einstein**

INTRODUCTION

Geometry is a very beautiful subject whose qualities of elegance, order and certainty have exerted a powerful attraction on the human mind for many centuries. The discoveries of Democritus and Archimedes about the volumes of cones and spheres (Sections 4 and 5 in this chapter) are among the most wonderful achievements of classical civilization. Also, the basic facts of geometry are absolutely essential for understanding many of the pure and applied sciences.

In spite of all this, most high school students emerge from their geometry courses with mixed feelings of confusion and relief. Why?

One of the reasons is that they have been ground down by complicated trivialities and offered little compensating insight into the geometric ideas that really matter. They have been bombarded with innumerable nit-picking definitions, and also with elaborate, boring “step-reason, step-reason” proofs of statements that in most cases are obvious to begin with. (At that stage, who can doubt the truth of such a statement as this: “Given any three points on a line, one is between the other two”? When asked to examine a proof, the natural reaction of an intelligent student is irritation and impatience, and he is right.) Their textbooks often seem to be written by the kind of person—we all know such people—who talks so much and says so little that we soon stop listening. All this tends to kill their interest in geometry long before they reach the meat of the subject.

The root of the problem is slavish adherence to the doctrine of Deductive Reasoning. This is the notion that knowledge is somehow not legitimate or genuine until it has been organized into an elaborate formal system of theorems that are carefully deduced from a small number of axioms or ‘self-evident truths’ stated at the beginning. Deductive Reasoning is an interesting idea that educated people ought to know something about, just as they should know something about representative government, the internal combustion engine, and other human inventions. It was very popular among philosophers and scientists of the 17th century, and was applied by them to physics,

GEOMETRY

ethics, and other unlikely subjects. Science shook off its grip 200 years ago, but geometry has continued to be strangled by this outmoded philosophical doctrine down to the present day.

In this chapter geometry is considered for its own sake and for the sake of its use as an indispensable tool in science and engineering, and *not* as a vehicle for teaching deductive reasoning. For us the purpose of proof is to remove doubt and convey insight, not to belabor the obvious. This point of view produces a gain in efficiency so great that almost everything of importance in plane and solid geometry can be said in about a dozen pages, with full explanations and proofs.* I have tried to include all the necessary facts and to omit everything else, however interesting or tempting it might be. (Sections 1 to 5 conform to this standard, but in Appendices E and F, I yielded to temptation and added a few delicious items just for the fun of it.) It should be added that no definitions are provided for such familiar geometric objects as triangles, parallel lines, circles, cones, spheres, and the like. This chapter is not intended to teach geometry to someone who knows absolutely nothing about it, but rather to review and clarify the main ideas of the subject; and if definitions are needed, they can be found in almost any dictionary.

I offer this bit of advice to the student. My explanations are deliberately very concise, with few words wasted. Also, much of the burden of the exposition is carried by the figures. Passive reading therefore will not do. If you wish to understand, it is necessary to read actively and carefully, thinking all the time, constantly asking *why?*—and constantly struggling to find an answer.

*I use the word "proof" to mean an argument that I hope my intended audience will find convincing. A few mathematicians may object to this relativistic attitude. However, since students differ from logicians in their power of skepticism, and logicians differ among themselves from one generation to the next, it seems unlikely that any fixed, unalterable, absolute meaning can possibly be attached to the concept of proof. What a proof is depends on who and when you are.

1. TRIANGLES

(a) SUM OF ANGLES.

The customary unit of measure for angles is the degree. One degree (1°) is one-ninetieth of a right angle (see Fig. 1). If a transversal (a transverse line) is drawn across a pair of parallel lines (Fig. 2), then corresponding angles are equal and alternate interior angles are equal, as shown. The

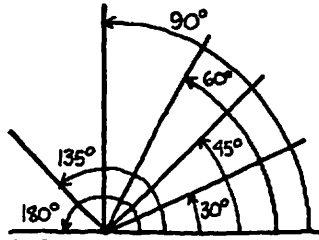


FIG. 1

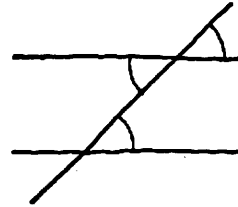
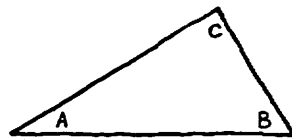


FIG. 2

sum of the angles in any triangle equals 180° (Fig. 3). This can be proved at once by inspecting the diagram shown in Fig. 4. As a direct consequence, we see that the sum of the acute angles



$$A + B + C = 180^\circ$$

FIG. 3

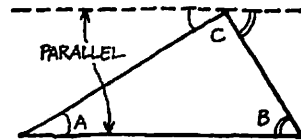


FIG. 4

in a right triangle equals 90° (Fig. 5). Also, in any triangle an exterior angle equals the sum of the opposite interior angles (Fig. 6).

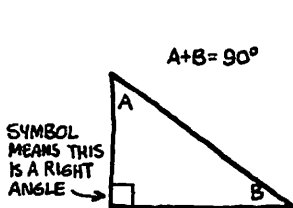


FIG. 5

SYMBOL
MEANS THIS
IS A RIGHT
ANGLE

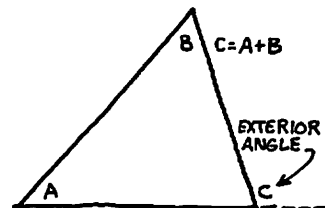


FIG. 6

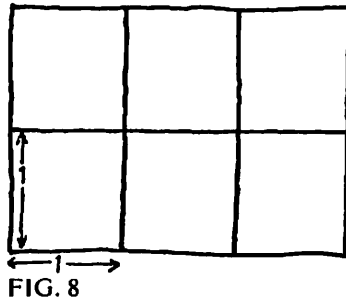
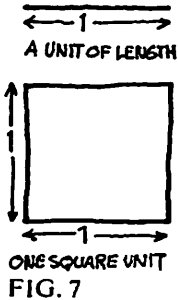
EXTERIOR
ANGLE

(b) AREA.

All ideas about area begin in this way: select a unit of length, draw a square whose side is this unit, and define the area of the square to be *one*

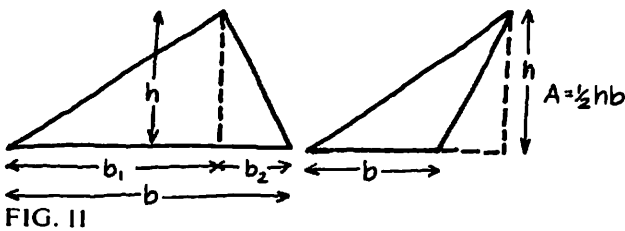
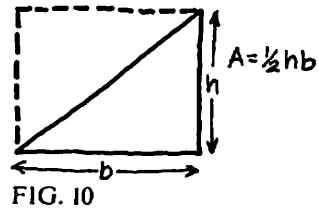
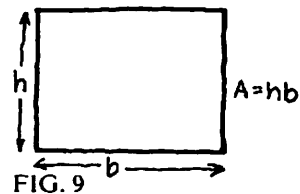
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square unit (Fig. 7). The rectangle shown in Fig. 8 has height 2 and base 3. The internal horizontal

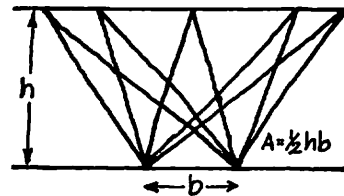


and vertical lines divide the rectangle into 6 squares each of area 1, so the area of the rectangle is evidently 6 square units. The fact that $6 = 2 \cdot 3$ suggests that the area A of an arbitrary rectangle of height h and base b should be defined by the formula $A = hb$ (Fig. 9). By Fig. 10, the area of a right triangle of height h and base b is given by

$A = \frac{1}{2}hb$. Since any triangle can be viewed as the sum or the difference of two right triangles (Fig. 11), for each of which this formula is valid, the formula is valid for all triangles. [Thus, in the triangle on the left in Fig. 11, b_1 and b_2 are the bases of the two right triangles and $A = \frac{1}{2}hb_1 + \frac{1}{2}hb_2 = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}hb$.] In Fig. 12 the two horizontal



lines are understood to be parallel, so all the triangles shown have the same height and the same base, and therefore the same area. We can express this in a different way by saying that if the base of the triangle is held fixed on the lower line and the upper vertex is moved back and forth along the upper line, then the area of the triangle does not change.



(c) SIMILARITY.

Roughly speaking, two triangles are similar if they have the same shape but different sizes, that is, if one is a magnified version of the other. The precise meaning of similarity for triangles is that their corresponding angles must be equal; and this implies that the ratios of their corresponding sides must also be equal, as shown in Fig. 13. The first of these equations ($\frac{a}{d} = \frac{b}{e}$) can be written in the equivalent form

$$\frac{a}{b} = \frac{d}{e}.$$

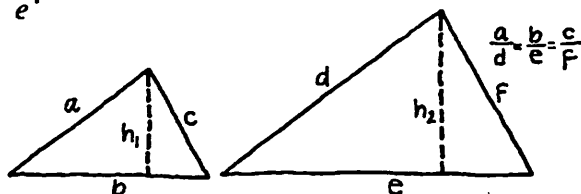


FIG. 13

In words: if two triangles are similar then the ratio of any two sides of one triangle equals the ratio of the corresponding sides of the other. By part (a) above, two triangles will necessarily be similar if two pairs of corresponding angles are equal. Finally, the ratio of the areas of two similar triangles equals the ratio of the squares of any pair of corresponding sides. [To see why this is true, notice in Fig. 13 that if $A_1 = \frac{1}{2}h_1b$ and $A_2 =$

$\frac{1}{2}h_2e$ then

$$\frac{A_1}{A_2} = \frac{\frac{1}{2}h_1b}{\frac{1}{2}h_2e} = \left(\frac{h_1}{h_2}\right)\left(\frac{b}{e}\right) = \left(\frac{b}{e}\right)\left(\frac{b}{e}\right) = \frac{b^2}{e^2}.$$

(d) PYTHAGOREAN THEOREM.

In its purely geometric form, this famous and indispensable theorem states that in any right triangle, if squares are constructed on the hypotenuse and the two legs, then the square on the hypotenuse equals the sum of the squares on the legs (Fig. 14). In a more algebraic version, it says that the square of the hypotenuse equals the sum of the squares of the legs. This is illustrated on the

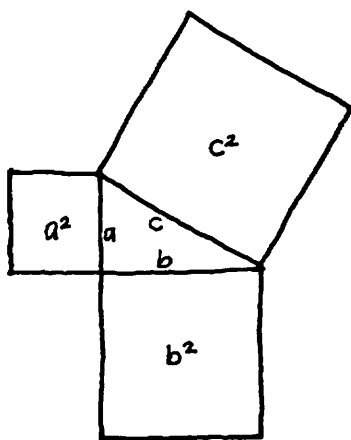


FIG. 14

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left in Fig. 15, and is proved on the right. [The proof is carried out by inserting four replicas of the triangle in the corners of a square of side $a + b$. The first equation on the right says that the area of the large square equals the area of the four triangles plus the area of the small inner square. Why is the inner quadrilateral a square?]

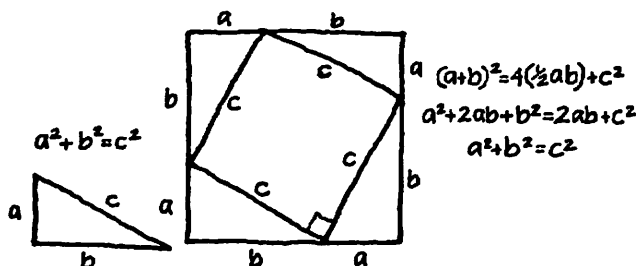


FIG. 15

2. CIRCLES

(a) THE NUMBER π .

The Greek letter π (pronounced 'pie') denotes the ratio of the circumference of a circle to its diameter (Fig. 16):

$$\pi = \frac{c}{2r}, \quad \text{so } c = 2\pi r.$$

The number π is known to be irrational. This means that it cannot be expressed exactly as a fraction or as a terminating decimal. However, its numerical value can be calculated in various ways to any specified degree of accuracy. This value is approximately 3.14, or even more accurately, 3.14159. The fraction $\frac{22}{7}$ is also a good approximation, better than 3.14 but not as good as 3.14159.*

(b) AREA.

Our purpose here is to understand the formula $A = \pi r^2$ for the area A of a circle in terms of its radius r .

*The student who wishes to learn more about the rich history of this fascinating number should read Isaac Asimov's article "A Piece of Pi," reprinted in his book, *Asimov on Numbers* (Doubleday, 1977). Even better is Petr Beckmann's book, *A History of Pi* (Golem Press, Boulder, Colo., 1971).

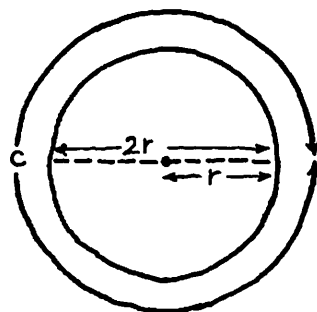


FIG. 16

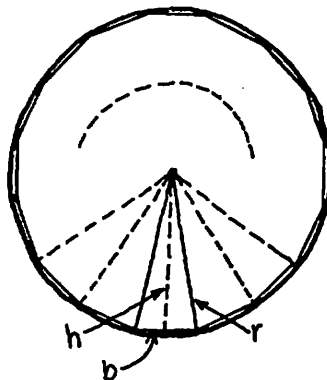


FIG. 17

PRECALCULUS MATHEMATICS

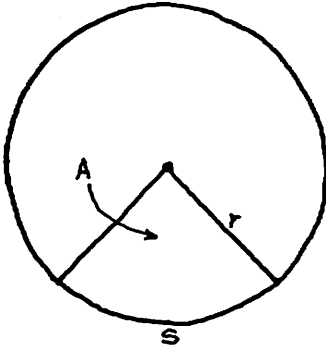


FIG. 18

INTERCEPTED ARC-LEN
AND SECTOR-AREA
ARE PROPORTIONAL.

Suppose that the circle has inscribed in it a regular polygon with a large number of sides (Fig. 17). Each of the small triangles shown in the figure has area $\frac{1}{2}hb$, and the sum of these areas equals the area of the polygon, which is approximately equal to the area of the circle. If p denotes the perimeter of the polygon, we see that

$$A_{\text{polygon}} = \frac{1}{2}hb + \frac{1}{2}hb + \cdots + \frac{1}{2}hb$$

$$= \frac{1}{2}h(b + b + \cdots + b) = \frac{1}{2}hp.$$

Let c be the circumference of the circle. Then, as the number of sides of the polygon increases, h approaches r (this is symbolized by writing $h \rightarrow r$, where the arrow means 'approaches'), $p \rightarrow c$, and therefore

$$A_{\text{polygon}} = \frac{1}{2}hp \rightarrow \frac{1}{2}rc = \frac{1}{2}r(2\pi r) = \pi r^2.$$

This establishes the formula stated above.

To find the area of a sector (Fig. 18), we use the observation that the ratio of this area to the total area of the circle equals the ratio of the intercepted arc s to the complete circumference:

$$\frac{A}{\pi r^2} = \frac{s}{2\pi r}, \text{ so } A = \frac{1}{2}rs.$$

This fact is easy to remember, since the area of the sector is exactly what it would be if the sector were a triangle with height r and base s .

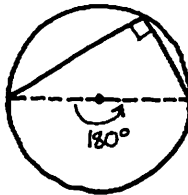


FIG. 19

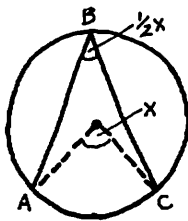


FIG. 20

(c) INSCRIBED ANGLES.

Fig. 19 illustrates the important fact that an angle inscribed in a semicircle is necessarily a right angle. This is most easily understood as a special case of the more general fact (Fig. 20) that an angle inscribed in an arc ABC of a circle always equals one-half of the corresponding central angle. To see why this is true, we begin by considering the simplest special case, in which one side of the inscribed angle passes through the center of the circle (Fig. 21). Here we see that the central angle x is an exterior angle of the indicated isosceles triangle; this central angle therefore equals the sum of the base angles of the

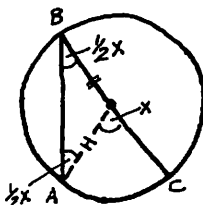


FIG. 21

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triangle, so each of these base angles equals $\frac{1}{2}x$.

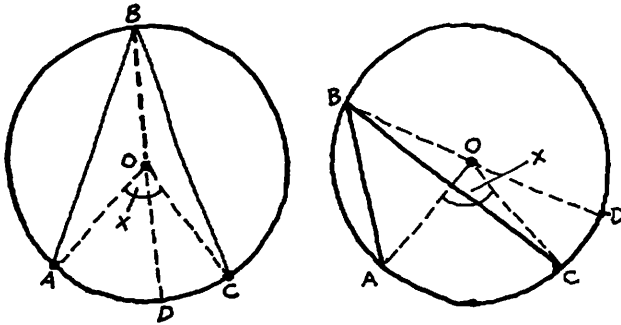


FIG. 22

The other two cases—in which the center of the circle lies inside or outside the inscribed angle (Fig. 22)—can easily be reduced to the case already considered, by drawing the diameter BD . [Thus, on the left in Fig. 22, $\angle ABC = \angle ABD + \angle DBC = \frac{1}{2}\angle AOD + \frac{1}{2}\angle DOC = \frac{1}{2}\angle AOC = \frac{1}{2}x$; and on the right, $\angle ABC = \angle ABD - \angle CBD = \frac{1}{2}\angle AOD - \frac{1}{2}\angle COD = \frac{1}{2}\angle AOC = \frac{1}{2}x$.]

A rather surprising conclusion can be drawn from this discussion: if the points A and C are held fixed, and B is moved to various positions on the circle, as shown in Fig. 23, then all of the corresponding inscribed angles are equal to one another.

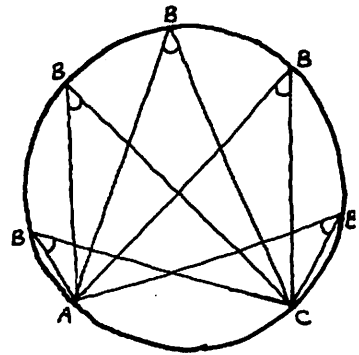


FIG. 23

3. CYLINDERS

All ideas about volume begin in this way: select a unit of length, consider a cube whose edge is this unit, and define the volume of this cube to be *one cubic unit* (Fig. 24). The rectangular box shown in Fig. 25 has height 3 and a rectangular base with sides 2 and 4. This box can be divided by horizontal and vertical planes into 3 layers of unit cubes in which each layer contains $2 \cdot 4 = 8$ cubes (in the figure we indicate the horizontal dividing planes). There are clearly $8 \cdot 3 = 24$ unit cubes altogether, so the volume of the box is 24 cubic units. The fact that the volume of this box is the area of the base times the height suggests that the volume V of an arbitrary rectangular box

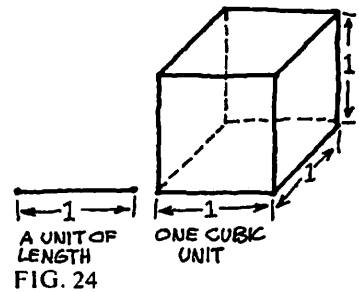


FIG. 24

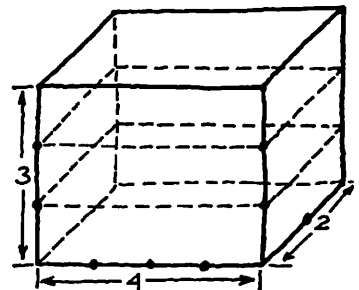


FIG. 25

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with height h and area of base B should be defined by the formula $V = Bh$ (Fig. 26). Similarly, the

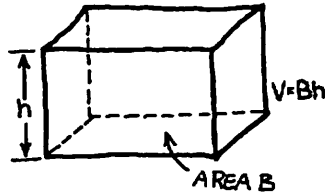


FIG. 26

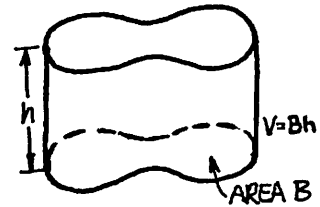


FIG. 27

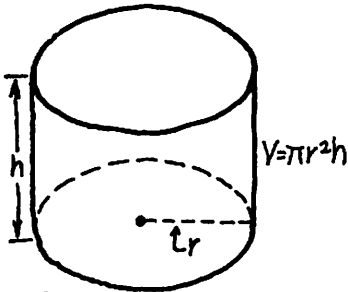


FIG. 28

volume of any solid with vertical walls and horizontal base and top (Fig. 27) is defined to be the area of the base times the height. In particular, the volume of a cylinder (understood to be a right circular cylinder) with height h and radius of base r (Fig. 28) is $V = \pi r^2 h$, since the area of the base is πr^2 .

Suppose that the top and bottom are removed from a cylinder and that its lateral surface is opened by a vertical cut and unrolled into a rectangle (Fig. 29). It is easy to see that the lateral area of the cylinder is the area of this rectangle, $2\pi r h$, and that the total area is $2\pi r h + 2\pi r^2$.

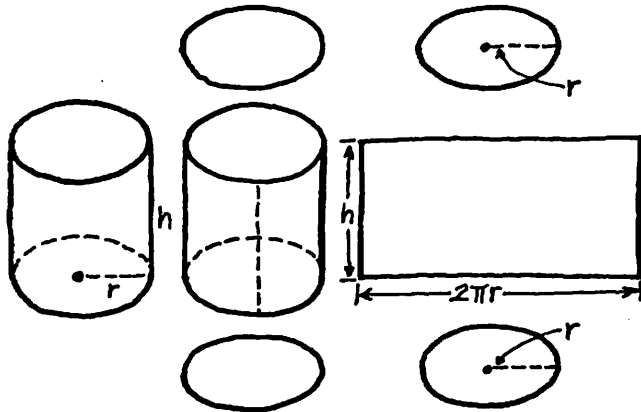


FIG. 29

4. CONES

Consider a cone (understood to be a right circular cone) with height h , radius of base r , and slant height s , as shown in Fig. 30. The fundamental facts about cones are the formulas stated in Fig. 30. The volume formula (the volume equals

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one-third the area of the base times the height, or equivalently, one-third the volume of the circumscribing cylinder) is difficult, and we discuss it in part (b) below. The lateral area formula is easier.

(a) LATERAL AREA.

The formula for the lateral area is proved in Fig. 31 by cutting the lateral surface of the cone down a generator and unrolling this surface into a sector of a circle.* This formula is easy to remember by thinking of the lateral surface as swept out by revolving a generator about the axis: the lateral area equals the length of this generator multiplied by the distance traveled by its midpoint, $s \cdot 2\pi\left(\frac{1}{2}r\right) = \pi rs$, as suggested in Fig. 30. It is also useful to know that the lateral area of a frustum of a cone (Fig. 32) equals the length of a generator multiplied by the distance traveled by its midpoint. This is proved in the figure.

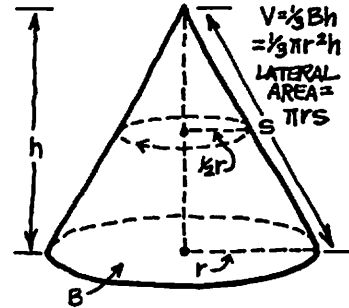


FIG. 30

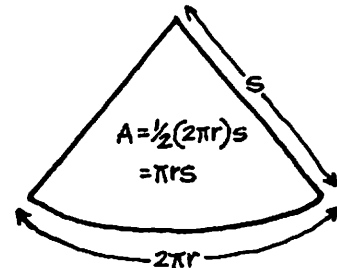


FIG. 31

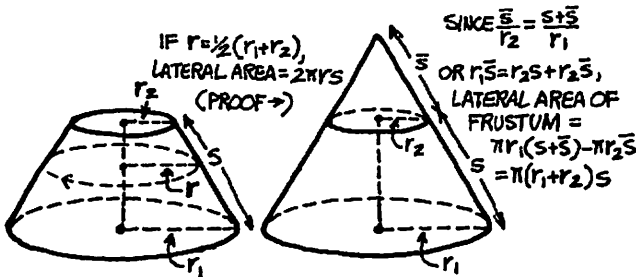


FIG. 32

(b) VOLUME.

In Fig. 33 we consider the cone shown in Fig. 30. Our purpose is to establish the volume formula $V = \frac{1}{3} Bh$, in particular to understand where the factor $\frac{1}{3}$ comes from. In the base of this cone we inscribe a regular polygon with n sides, where n is some large number (in the figure, $n = 8$). Using this polygon as a base, we construct

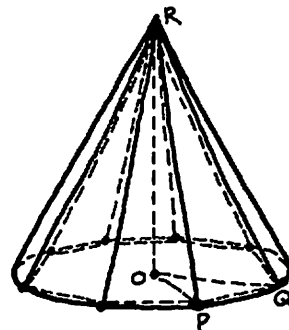
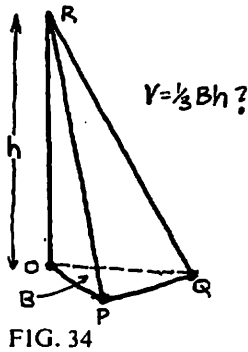


FIG. 33

*A *generator* of a cone is a straight line down the side, joining the vertex to a point on the circumference of the base.



a pyramid whose vertex is the vertex of the cone. The volume of the cone is the limiting value approached by the volume of the pyramid as n increases. To prove the volume formula for the cone, it therefore suffices to show that the volume of the pyramid is one-third the area of its base times its height. Since the pyramid can be divided into n congruent pyramids of the type shown in Fig. 34, it suffices to show that the volume formula is valid for these special pyramids. This we now do.

On the left in Fig. 35 we show the pyramid in

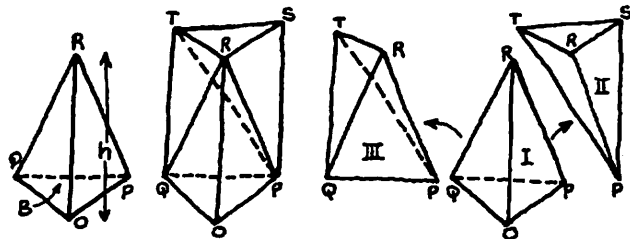


FIG. 35

Fig. 34 in a slightly different position. On the base OPQ we construct a prism with height h and base area B (Fig. 35, center). This prism can be divided into three pyramids as shown on the right in the figure. Pyramids I and II have height h and triangular bases OPQ and RST of equal area, so they have equal volumes. Pyramids II and III have the same height (the distance from R to the plane $PQST$) and triangular bases PST and PQT of equal area, so they also have equal volumes. This argument shows that all three pyramids have the same volume, so the volume of each is one-third the volume Bh of the prism. This establishes the volume formula in Fig. 34, and with it the volume formula for the cone as stated in Fig. 30.*

*The argument given here makes use of the fact that two pyramids with the same height and triangular bases with the same area have equal volumes. This fact is proved in Appendix D by means of Cavalieri's Principle as stated in the next section.

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5. SPHERES

Our purpose is to establish the formulas stated in Fig. 36 for the volume V and surface area A of a sphere. These are profound facts and require ingenious methods.

(a) CAVALIERI'S PRINCIPLE.

Consider a rectangular solid (Fig. 37, left) consisting of a stack of thin rectangular cards, all with the same dimensions. The shape of this stack can easily be altered without changing its volume, by gently pushing at it horizontally (Fig. 37, right). The volume before is clearly the same as the volume after, since each card in the stack is unchanged except in its position relative to nearby cards. Next, consider two solids with different shapes but the same height (Fig. 38), made up of equal numbers of thin cards. If we assume that each card in one stack has the same face area as the corresponding card in the other stack, regardless of the different shapes of these cards, then it seems reasonable to conclude that the two solids have the same volume. These remarks about the volumes of solids consisting of stacks of thin cards suggest a very powerful principle in the theory of volumes. This principle was first formulated by the Italian mathematician whose name it bears. *Cavalieri's Principle* states that if two solids have the property that every plane parallel to a fixed plane intersects them in cross-sections having equal areas, then the two solids have the same volume (Fig. 39).

(b) THE VOLUME FORMULA.

We find the volume of a sphere of radius r by comparing the sphere with the following solid (Fig. 40): consider a cylinder with base radius r

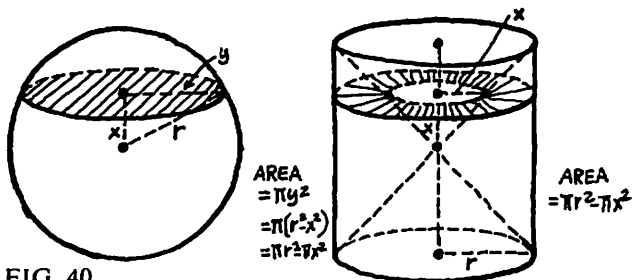


FIG. 40

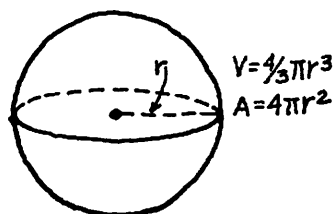


FIG. 36

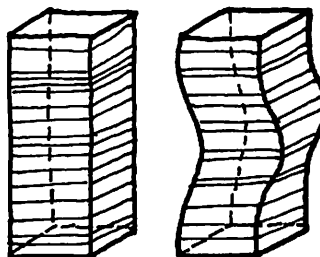


FIG. 37

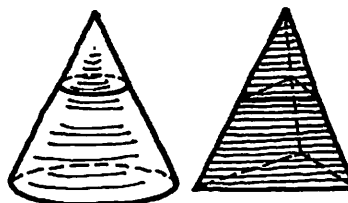


FIG. 38

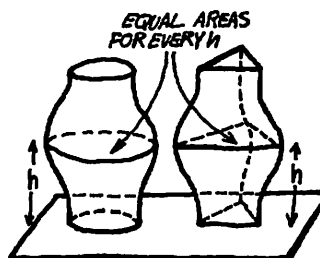


FIG. 39

and height $2r$; the comparison solid is what remains of this cylinder after the removal of the two cones shown in the figure, that is, it is the cylinder with two conical hollows on the ends. If we calculate the areas of corresponding cross-sections of these solids, as indicated in the figure, we find that they are equal. By Cavalieri's Principle, the solids have equal volumes. The volume V of the sphere, being equal to the volume of the cylinder minus the volumes of the two cones, is therefore given by the formula

$$\begin{aligned} V &= \pi r^2(2r) - 2\left(\frac{1}{3}\pi r^2 \cdot r\right) \\ &= 2\pi r^3 - \frac{2}{3}\pi r^3 \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

(c) THE SURFACE AREA FORMULA.

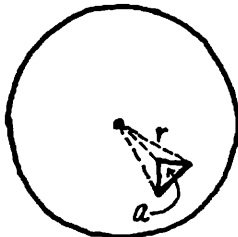


FIG. 41

We find the surface area A of a sphere of radius r by dividing the solid sphere into a large number of small "pyramids." Imagine that the surface of the sphere is divided into a large number of tiny "triangles," as suggested in Fig. 41. These are not actually triangles, since there are no straight line segments on the surface of a sphere. However, being very small, they are nearly triangles. Let each such "triangle" be used as the base of a "pyramid" whose vertex is the center of the sphere. If a is the area of the base of our tiny "pyramid," and r is its height, then its volume v is given by $v = \frac{1}{3}ar$. If we add these equations for all such "pyramids," filling the solid sphere, we see that the volume V and surface area A of the sphere are connected by the equation

$$V = \frac{1}{3}Ar.$$

Since we know that $V = \frac{4}{3}\pi r^3$, we have

$$\frac{4}{3}\pi r^3 = \frac{1}{3}Ar,$$

and therefore

$$A = 4\pi r^2.$$

GEOMETRY

APPENDIX A. THE MAIN FORMULAS OF GEOMETRY

The formulas stated here express the main content of Sections 1 to 5.

TRIANGLES (Figs. 42, 43)

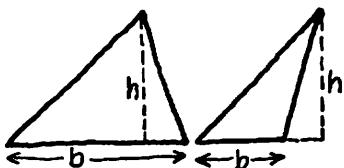


FIG. 42

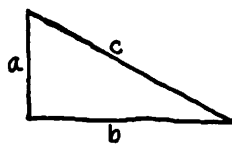


FIG. 43

$$\text{area } A = \frac{1}{2}hb$$

$$\text{Pythagorean theorem: } a^2 + b^2 = c^2$$

CIRCLES (Fig. 44)

$$\text{circumference } c = 2\pi r$$

$$\text{area } A = \pi r^2$$

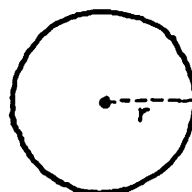


FIG. 44

CYLINDERS (Fig. 45)

$$\text{lateral area } A = 2\pi rh$$

$$\text{volume } V = \pi r^2 h$$

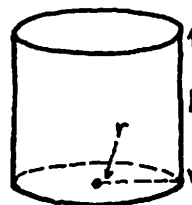


FIG. 45

CONES (Fig. 46)

$$\text{lateral area } A = \pi rs$$

$$\text{volume } V = \frac{1}{3}\pi r^2 h$$

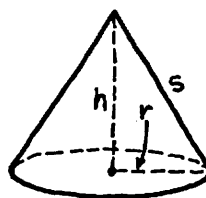


FIG. 46

SPHERES (Fig. 47)

$$\text{surface area } A = 4\pi r^2$$

$$\text{volume } V = \frac{4}{3}\pi r^3$$

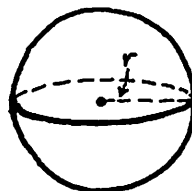


FIG. 47

APPENDIX B. A NUMBER OF EXERCISES, SOME EASY AND SOME HARD

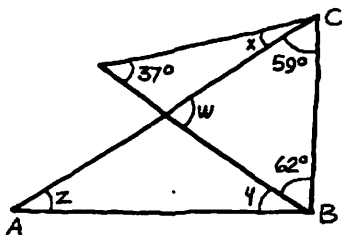
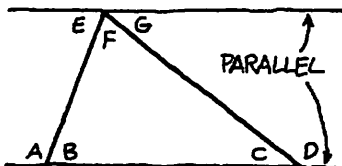
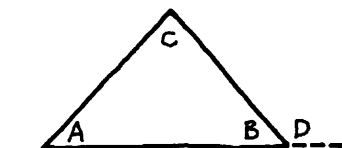
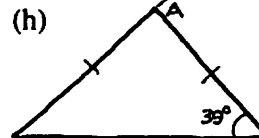
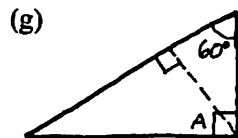
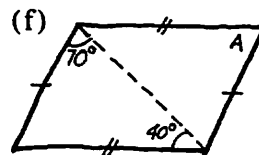
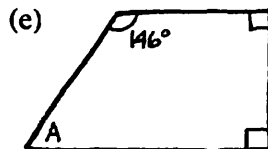
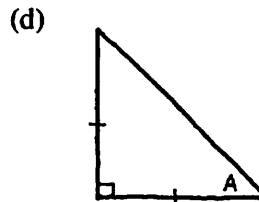
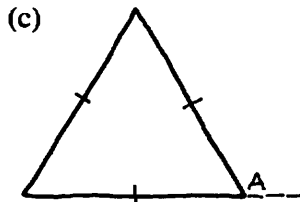
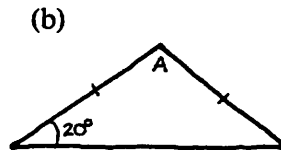
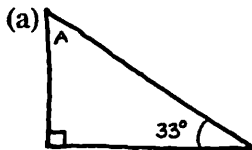
78 PROBLEMS IN 5 SECTIONS.

The harder (and therefore more interesting) exercises are marked with an asterisk (*). The exercises with full solutions given in Appendix C are marked with two asterisks (**).

SECTION 1

1. Find the angle A in each of the following figures:

ANGLES OPPOSITE
EQUAL SIDES ARE
EQUAL.



2. In each case find the required angle:

- (a) if $A = 40^\circ$ and $C = 100^\circ$, $D = ?$;
- (b) if $A = 50^\circ$ and $B = 30^\circ$, $C = ?$;
- (c) if $A = 50^\circ$ and $D = 140^\circ$, $C = ?$

3. In each case find the angles not given:

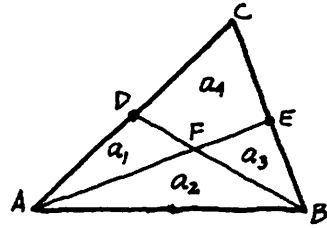
- (a) $A = 150^\circ$, $C = 40^\circ$;
- (b) $B = 60^\circ$, $C = 65^\circ$.

**4. In the figure, $AB \perp BC$ (the symbol \perp means "is perpendicular to"). Find the angles x , y , z , w .

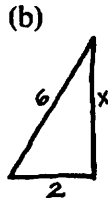
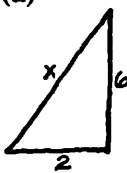
- 5. What is the height of a rectangle whose area is 40 square inches and whose base is 8 inches?
- 6. If the base of a triangle is 9 inches and its area is 45 square inches, what is its height?

GEOMETRY

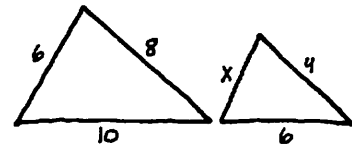
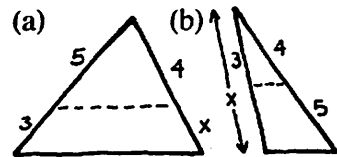
7. The height and base of a triangle are equal and its area is 32 square inches. Find the height and base.
- *8. In the triangle ABC , D and E are the midpoints of AC and BC . The segments AE and BD intersect at F . Show that regions a_2 and a_4 have equal areas.



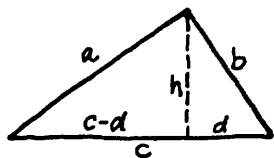
9. A quarter (a 25¢ piece) is $\frac{3}{4}$ inch in diameter, and when placed 7 feet from the eye will just block out the disc of the moon. If the diameter of the moon is 2160 miles, how far is the moon from the earth?
10. A man 6 feet tall stands at the foot of a flagpole. If the shadow of the man and the pole are 4 feet and 40 feet in length, how tall is the pole?
11. In each figure the dotted line is parallel to a side of the triangle. Find x .
12. Find x and y in the adjoining similar triangles.
13. The areas of two similar triangles are 25 and 16. If the perimeter of the first is 15, find the perimeter of the second.
14. Two equilateral triangles have sides of 4 inches and 6 inches. What is the ratio of their perimeters? Of their areas? Of their heights?
15. Find x in each of the following right triangles:



16. A 20-foot ladder leans against a wall with its foot 6 feet from the wall. A man stands on a rung which is 12 feet from the bottom of the ladder. How far is the man from the wall and from the ground?
17. What is the diagonal of a square whose side is a ?
18. What is the side of a square whose diagonal is a ?



PRECALCULUS MATHEMATICS



19. Find the area of an equilateral triangle whose side is a .
- *20. Heron's formula states that the area of a triangle with sides a, b, c is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$, where the number $s = \frac{1}{2}(a+b+c)$ is called the semiperimeter. Prove this formula by verifying the following steps:

(a) $A = \frac{1}{2}hc$;

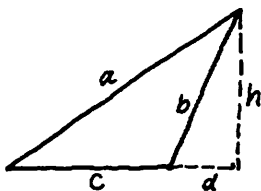
(b) $a^2 = b^2 + c^2 - 2cd$;

(c) $d = \frac{b^2 + c^2 - a^2}{2c}$;

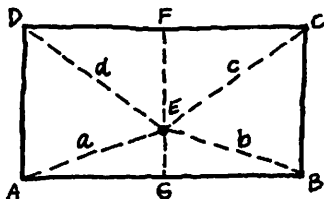
$$\begin{aligned} \text{(d) } h^2 &= b^2 - d^2 = b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2} \\ &= \frac{(a+b+c)(b+c-a)(a+b-c)(a-b+c)}{4c^2} \\ &= \frac{2s(2s-2a)(2s-2c)(2s-2b)}{4c^2} \\ &= \frac{4s(s-a)(s-b)(s-c)}{c^2}; \end{aligned}$$

(e) $h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$;

(f) $A = \sqrt{s(s-a)(s-b)(s-c)}$.



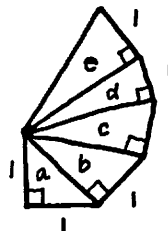
These steps require changes for a triangle of the shape indicated at the left. Provide these changes, and in this way show that the area formula is valid without any restrictions at all.



21. Use Heron's formula and one other method to find the area of a right triangle with sides 3, 4, 5.
22. Apply Heron's formula to verify the result of Exercise 19.
23. Find the hypotenuse of a right triangle whose legs are (a) 3, 4; (b) 5, 12; (c) 6, 8; (d) 7, 24; (e) 8, 15.
24. The hypotenuse of a right triangle is 15 and one leg is 12. What is the other leg?
- **25. If E is any interior point of the indicated rectangle, show that $a^2 + c^2 = b^2 + d^2$.

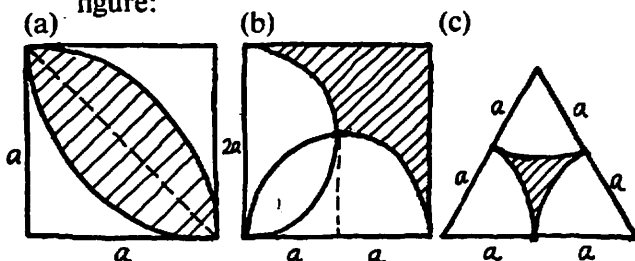
GEOMETRY

26. In the adjoining figure, find a, b, c, d, e .
- *27. Show that in any parallelogram the sum of the squares of the diagonals equals the sum of the squares of the four sides.
28. A side of one square equals a diagonal of a second square. Find the ratio of the area of the larger square to that of the smaller square.
29. A side of one equilateral triangle equals the height of a second equilateral triangle. Find the ratio of the perimeter of the larger triangle to that of the smaller.
30. Show that in a 30° - 60° right triangle the altitude on the hypotenuse divides the hypotenuse into segments whose lengths have the ratio $1/3$.



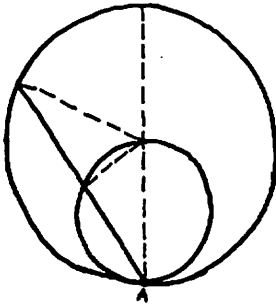
SECTION 2

- The diameter of one circle equals the radius of a second circle. Find the ratio of their areas.
 - The ratio of the areas of two circles of radii R and r is $2/1$. What is the ratio R/r of their radii?
 - Find the area of a sector of a circle of radius 10 whose central angle is (a) 60° ; (b) 90° ; (c) 48° .
 - Two concentric circles have circumferences 30π and 40π . Find the area of the ring-shaped region between them.
 - Show that the area of the ring-shaped region between two concentric circles equals the area of a circle whose diameter is a chord of the outer circle which is tangent to the inner circle.
- *6. Find the area of the shaded part of each figure:

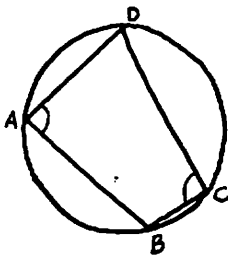


PRECALCULUS MATHEMATICS

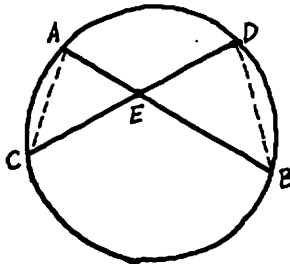
7. A goat is tied to the corner of a shed 12 feet long and 10 feet wide. If the rope is 15 feet long, over how many square feet can the goat graze?
8. The earth is approximately 93 million miles from the sun. Assuming that the earth's orbit is circular, approximately how far does the earth move along its orbit in each second? Use the approximation $\pi = \frac{22}{7}$.



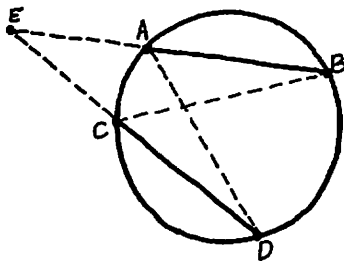
9. Consider two circles with the second internally tangent to the first at a point A and also passing through the center of the first. Show that every chord of the first circle which has A as an endpoint is bisected by the second circle.



10. If $ABCD$ is a quadrilateral inscribed in a circle, show that the opposite angles A and C are supplementary ($A + C = 180^\circ$).



- **11. AB and CD are two chords in a circle, and they intersect at a point E . Show that the product of the segments of one chord equals the product of the segments of the other chord, that is, that $AE \cdot BE = CE \cdot DE$. Hint: use similar triangles.



12. If AB and CD are two chords of a circle that have been extended to intersect at an external point E , show that $AE \cdot BE = CE \cdot DE$.

GEOMETRY

SECTION 3

1. Find the volume and the total surface area of a rectangular box with edges 4, 5 and 6 feet.
2. Find the length of the diagonal joining two opposite vertices of the box in the preceding exercise.
3. What is the volume of a cube whose total surface area is 150 square inches?
4. A cylinder is 7 inches high and the radius of its base is 4 inches. Use the approximation $\pi = \frac{22}{7}$ and calculate (a) its volume; (b) its lateral area; (c) its total area.
- **5. If the radius of the base of a cylinder is doubled and its height is tripled, by what number is the volume multiplied?
6. Find the volume of a cylinder if the radius of its base is one-third its height h .
7. In a certain cylinder the lateral area is half the total area. How is the height h related to the radius r of the base?

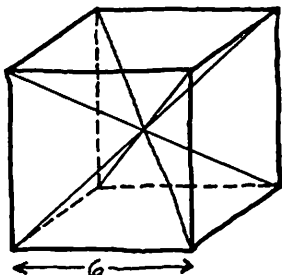
LATERAL AREA
"SIDE"

SECTION 4

1. Find the volume of a cone 28 feet high whose base has diameter 12 feet. Use the approximation $\pi = \frac{22}{7}$.
2. Find the height of a cone whose volume is 484 cubic inches and whose base has radius 7 inches. Use the approximation $\pi = \frac{22}{7}$.
- **3. The height of a cone equals the radius r of its base. Show that the volume V is given by the formula $V = \frac{\sqrt{2}}{6}rA$ where A is the lateral area.
- *4. The height of a cone is h . A plane parallel to the base intersects the axis at a certain point. How far from the vertex must this point be if the plane divides the lateral area into two equal parts? If the plane divides the volume into two equal parts?

PRECALCULUS MATHEMATICS

5. The height of a cone is h and the radius of its base is r . If r is halved, how must h be changed to keep the volume unchanged?
6. A plane parallel to the base of a cone bisects the axis. What is the ratio of the volume of the original cone to the volume of the frustum formed in this way?
7. The Great Pyramid of Egypt originally had a square base 755 feet on a side and was 481 feet high. Compute its volume.
8. The adjoining cube consists of six identical pyramids. Find the volume of one of these pyramids in two different ways.
9. A conical tent is made by using a semi-circular piece of canvas of radius 8 feet. Find the height of the tent and the number of cubic feet of air inside.



SECTION 5

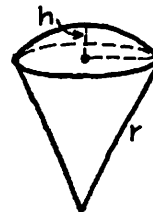
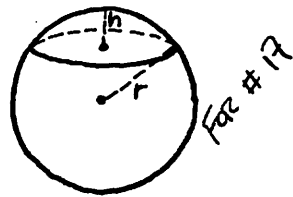
1. Find the volume of a sphere whose diameter is 6.
2. Find the radius of a sphere whose volume is 2304π .
3. Find the area of a sphere if the circumference of a great circle is 40π . (A great circle is the intersection of the surface of a sphere with a plane through its center.)
4. Find the radius of a sphere whose area is 36π .
5. The radius of the earth is approximately 4000 miles and the area of the continental United States is approximately 3,000,000 square miles. What percentage of the total area of the earth does this area represent?
Use the approximation $\pi = \frac{22}{7}$.
6. A cylinder is circumscribed about a sphere. Find the ratio of the volume of the cylinder to the volume of the sphere.
7. A cylinder is circumscribed about a sphere. Show that the area of the sphere equals the lateral area of the cylinder.
- *8. If the radius of a certain sphere is increased by 6, its area is multiplied by 9. Find the radius of the sphere.

GEOMETRY

9. A cylinder with height 6 and radius 4 is inscribed in a sphere. Find the area and volume of the sphere.
10. A cylinder is circumscribed about a sphere. A cone has the same base and height as the cylinder. Find the ratio of the total area of the cylinder to that of the sphere and of the cone.
- *11. An equilateral triangle and a square are inscribed in a circle, with a side of the triangle being parallel to a side of the square. The entire figure is revolved about that altitude of the triangle which is perpendicular to a side of the square. Find the ratio of the area of the sphere to the total area of the cylinder, and the ratio of the total area of the cylinder to the total area of the cone.
- **12. A sphere is inscribed in a cone. The slant height of the cone equals the diameter of its base. If the height of the cone is 9, find the area of the sphere.
- *13. The ratio of the volumes of two spheres is $27/343$ and the sum of their radii is 10. What is the radius of the smaller sphere?
14. A sphere is circumscribed about a cube. Find the ratio of the volume of the cube to the volume of the sphere.
15. A cylinder is circumscribed about a sphere. Show that the ratio of their volumes equals the ratio of their total areas.
16. Two spheres of radii 3 inches and 5 inches rest on a table and touch one another. How far apart are the points at which they touch the table?
- *17. Use Cavalieri's Principle to find the volume of a spherical segment of one base and thickness h if the radius of the sphere is r .
- *18. In the preceding exercise, find the volume of the spherical sector (the solid shown on the right, resembling a filled ice cream cone). Use this result, and a comparison of areas and volumes, to show that the area of the curved surface on top of the sector is $2\pi rh$.

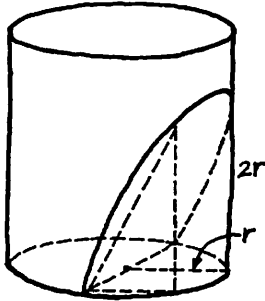
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PRECALCULUS MATHEMATICS

- *19. A spherical ring is the solid that remains after removing from a solid sphere of radius r a cylindrical boring whose axis passes through the center of the sphere. If h is the height of the ring, use the result of Exercise 17 to show that the volume of the ring is $\frac{\pi h^3}{6}$. (Notice how remarkable it is that this volume depends only on h , and not on the radius r of the sphere.)



- *20. A cylindrical wedge is the solid cut from a cylinder by a tilted plane passing through a diameter of the base. Apply Cavalieri's Principle to find the volume of such a wedge if its height is $2r$, where r is the radius of the base and the height of the cylinder is $> 2r$. (Use as a comparison solid a rectangular box having edges $r, r, 2r$ with two square pyramids removed, where the pyramids have the square ends of the box as bases and common vertex at the center of the box. Stand the box on one of its square ends and place the wedge so that the bounding diameter of its base is vertical.)

APPENDIX C. THE ANSWERS TO THE EXERCISES, WITH FULL SOLUTIONS FOR A SELECTED FEW

SECTION 1

1. (a) 57° ; (b) 140° ; (c) 120° ; (d) 45° ; (e) 34° ; (f) 70° ; (g) 60° ; (h) 78° .
2. (a) $D = 140^\circ$; (b) $C = 100^\circ$; (c) $C = 90^\circ$.
3. (a) $B = 30^\circ, D = 140^\circ, E = 30^\circ, F = 110^\circ, G = 40^\circ$; (b) $A = 120^\circ, D = 115^\circ, E = 60^\circ, F = 55^\circ, G = 65^\circ$.
4. Since the sum of the angles of a triangle equals 180° , we have $(x + 59^\circ) + 62^\circ + 37^\circ = 180^\circ$, and therefore $x = 180^\circ - (59^\circ + 62^\circ + 37^\circ) = 180^\circ - 158^\circ = 22^\circ$. Since a right angle equals 90° , $y + 62^\circ = 90^\circ$ and therefore $y = 28^\circ$. Since the sum of the acute angles in a right triangle equals 90° , $z + 59^\circ = 90^\circ$ and $z = 31^\circ$. Finally, $w + 59^\circ + 62^\circ = 180^\circ$, so $w = 180^\circ - 121^\circ = 59^\circ$.
5. 5 inches. 6. 10 inches. 7. $h = b = 8$ inches.

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9. 241,920 miles. 10. 60 feet. 11. (a) $\frac{12}{5}$; (b) $\frac{27}{4}$.
12. $x = \frac{18}{5}$, $y = \frac{24}{5}$. 13. 12. 14. $\frac{2}{3}$, $\frac{4}{9}$, $\frac{2}{3}$.
15. (a) $2\sqrt{10}$; (b) $4\sqrt{2}$; (c) $4\sqrt{3}$.
16. $\frac{12}{5}$ feet from the wall, $\frac{6}{5}\sqrt{91}$ feet from the
ground. 17. $\sqrt{2}a$. 18. $\frac{1}{2}\sqrt{2}a$. 19. $\frac{1}{4}\sqrt{3}a^2$.
21. 6. 23. (a) 5; (b) 13; (c) 10; (d) 25; (e) 17.
24. 9. 25. By the Pythagorean theorem, we
see that $a^2 + c^2 = AG^2 + GE^2 + CF^2 + FE^2$
and $b^2 + d^2 = BG^2 + GE^2 + DF^2 + FE^2$. But
these expressions are equal, because $AG = DF$
and $BG = CF$.
26. $a = \sqrt{2}$, $b = \sqrt{3}$, $c = \sqrt{4}$, $d = \sqrt{5}$, $e = \sqrt{6}$.
28. 2. 29. $\frac{2}{3}\sqrt{3}$.

SECTION 2

1. $\frac{1}{4}$. 2. $R/r = \sqrt{2}$.
3. (a) $16\frac{2}{3}\pi$; (b) 25π ; (c) $13\frac{1}{3}\pi$. 4. 175π .
6. (a) $\frac{1}{2}a^2(\pi - 2)$; (b) $\frac{1}{2}a^2(6 - \pi)$;
(c) $\frac{1}{2}a^2(2\sqrt{3} - \pi)$. 7. $177\frac{1}{4}\pi$ square feet.
8. Approximately 18 miles. 11. The angles
 ACD and ABD are equal, because they are
inscribed in the same arc of the circle. The
angles CAB and CDB are equal for the same
reason. The angles AEC and DEB are evidently
equal. The triangles ACE and DBE are there-
fore similar, and consequently $\frac{AE}{DE} = \frac{CE}{BE}$ or
 $AE \cdot BE = CE \cdot DE$.

SECTION 3

1. Volume = 120 cubic feet, area = 148 square
feet. 2. $\sqrt{77}$ feet. 3. 125 cubic inches.
4. (a) 352 cubic inches; (b) 176 square inches;
(c) $276\frac{4}{7}$ square inches. 5. If the original radius
and height are r and h , then the new radius and
height are $2r$ and $3h$. The original volume is

PRECALCULUS MATHEMATICS

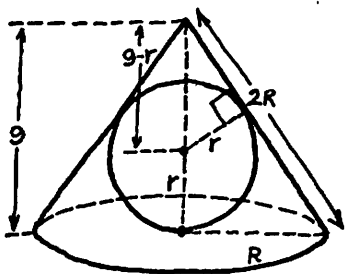
$\pi r^2 h$ and the new volume is $\pi(2r)^2(3h) = 12\pi r^2 h$, so the new volume is 12 times the original volume. 6. $\frac{\pi}{9} h^3$. 7. $h = r$.

SECTION 4

1. 1056 cubic feet. 2. $9\frac{3}{7}$ inches.
3. Since $r = h$, $A = \pi r \sqrt{r^2 + h^2} = \pi r \sqrt{2h^2} = \pi \sqrt{2} r h$, and $V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi r (r h) = \frac{1}{3} \pi r \left(\frac{A}{\pi \sqrt{2}} \right) = \frac{r A}{3 \sqrt{2}} = \frac{r A}{3 \sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{6} r A$.
4. $\frac{1}{\sqrt{2}} h$ for the lateral area, $\frac{1}{\sqrt[3]{2}} h$ for the volume.
5. h must be multiplied by 4. 6. 8/7.
7. 91,394,008 $\frac{1}{3}$ cubic feet. 8. 36.
9. Height = $4\sqrt{3}$ feet, volume = $\frac{64}{3} \sqrt{3} \pi$ cubic feet.

SECTION 5

1. 36π . 2. 12. 3. 1600π . 4. 3. 5. About 1.5%.
6. 3/2. 8. 3. 9. $A = 100\pi$, $V = 500\frac{\pi}{3}$.
10. $\frac{A_{\text{cyl}}}{A_{\text{sph}}} = \frac{3}{2} \cdot \frac{A_{\text{cyl}}}{A_{\text{cone}}} = \frac{6}{\sqrt{5} + 1}$.
11. $\frac{A_{\text{sph}}}{A_{\text{cyl}}} = \frac{A_{\text{cyl}}}{A_{\text{cone}}} = \frac{4}{3}$.



12. Draw a good picture of the situation, as shown. If r is the radius of the sphere and R is the radius of the base of the cone, then by using similar triangles we see that $\frac{9-r}{r} = \frac{2R}{R} = 2$, so $\frac{9}{r} - 1 = 2$, $\frac{9}{r} = 3$, $r = 3$, and therefore $A = 4\pi r^2 = 4\pi 9 = 36\pi$. The similar triangles mentioned are two right triangles with an acute angle in common, and the equation first written involves the ratio of the hypotenuse to the shorter leg in each. 13. 3. 14. $2\sqrt{3}/3\pi$. 16. $2\sqrt{15}$ inches.
17. $V = \pi h^2 \left(r - \frac{h}{3} \right)$. 18. $V = \frac{2}{3} \pi r^2 h$. 20. $\frac{4}{3} r^3$.

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APPENDIX D. TRIANGULAR PYRAMIDS WITH EQUAL HEIGHTS AND BASES

Our purpose is to fill a crucial gap in the argument of Section 4 (see the footnote in that section).

Consider a pyramid with vertex V and triangular base ABC in a horizontal plane (Fig. 48). Its height h is the perpendicular distance from V to the plane of ABC . Let the pyramid be cut by a second horizontal plane whose distance above the first plane is k , where $k < h$. This second plane intersects the pyramid in a triangle $A'B'C'$ which is similar to ABC . We assert that the areas of these triangles are related by the equation

$$\text{area } A'B'C' = \left(\frac{h-k}{h}\right)^2 \cdot \text{area } ABC.$$

[Proof: By similar triangles,

$$\begin{aligned} \frac{VC'}{VC} = \frac{VQ}{VP} = \frac{h-k}{h} \quad \text{and} \quad \frac{A'C'}{AC} &= \frac{VC'}{VC} \\ &= \frac{VQ}{VP} = \frac{h-k}{h}, \end{aligned}$$

so

$$\frac{\text{area } A'B'C'}{\text{area } ABC} = \left(\frac{A'C'}{AC}\right)^2 = \left(\frac{h-k}{h}\right)^2.]$$

Next, consider two pyramids with triangular bases in the same horizontal plane (Fig. 49). If they have the same base area B and the same height h , then they have the same volume. [Proof: By the above paragraph, horizontal cross-sections at height k have the same area B_k , where

$$B_k = \left(\frac{h-k}{h}\right)^2 B.$$

Cavalieri's Principle now implies that the pyramids have the same volume.]

APPENDIX E. CEVA'S THEOREM

It is not difficult to see that the angle bisectors of any triangle are concurrent, in the sense that they all pass through a single point (Fig. 50). Many facts about concurrent lines associated with triangles can be understood by means of a remarkable theorem discovered by the 17th century Italian mathematician Giovanni Ceva.

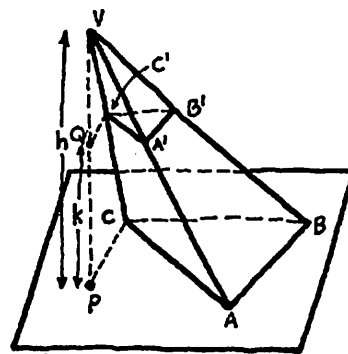


FIG. 48

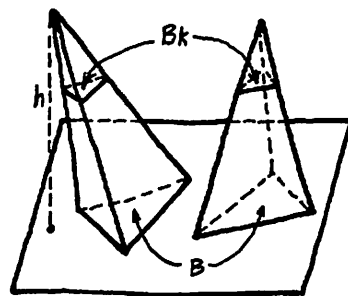


FIG. 49

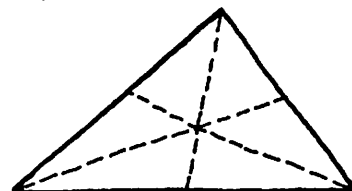


FIG. 50

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In order to formulate this theorem it is convenient to introduce the following term. A line segment joining a vertex of a triangle to a point on the opposite side is called a *cevian*. In each triangle of Fig. 51 we show three cevians AX , BY , CZ ; on the left these cevians are concurrent

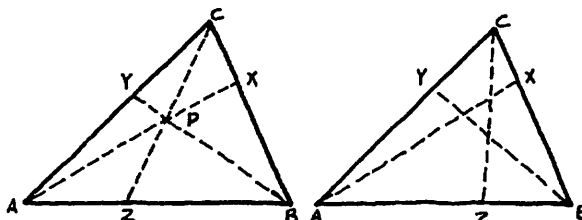


FIG. 51

at a point P , but on the right they are not concurrent. Ceva's theorem gives a criterion for concurrence in terms of the lengths of the six segments into which three such cevians divide the sides of the triangle.

CEVA'S THEOREM

Three cevians AX , BY and CZ of a triangle ABC are concurrent if and only if

$$\frac{AZ}{BZ} \cdot \frac{BX}{CX} \cdot \frac{CY}{AY} = 1.$$

Proof. We first assume that the cevians are concurrent at P , as shown on the left in Fig. 51. For convenience we denote the area of a triangle ABC by the symbol (ABC) . Since the areas of triangles with equal heights are proportional to their bases, we have

$$\begin{aligned} \frac{AZ}{BZ} &= \frac{(ACZ)}{(BCZ)} = \frac{(APZ)}{(BPZ)} = \frac{(ACZ) - (APZ)}{(BCZ) - (BPZ)} \\ &= \frac{(APC)}{(BPC)}. \end{aligned}$$

In the same way,

$$\frac{BX}{CX} = \frac{(APB)}{(APC)} \quad \text{and} \quad \frac{CY}{AY} = \frac{(BPC)}{(APB)}.$$

*In the middle step of this calculation we are essentially saying that if four numbers a , b , c , d satisfy the equation

$$\frac{a}{b} = \frac{c}{d}, \quad \text{then} \quad \frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d}.$$

This is easy to verify by cross-multiplication.

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By multiplying these three equations together, we obtain our conclusion,

$$\frac{AZ}{BZ} \cdot \frac{BX}{CX} \cdot \frac{CY}{AY} = \frac{(APC)}{(BPC)} \cdot \frac{(APB)}{(APC)} \cdot \frac{(BPC)}{(APB)} = 1.$$

To prove the other half of the theorem, we assume that

$$\frac{AZ}{BZ} \cdot \frac{BX}{CX} \cdot \frac{CY}{AY} = 1.$$

Let AX and BY intersect at P (Fig. 52) and let CP extended intersect AB at Z' . Then by the first part of the proof we know that

$$\frac{AZ'}{BZ'} \cdot \frac{BX}{CX} \cdot \frac{CY}{AY} = 1.$$

These two equations imply that

$$\frac{AZ'}{BZ'} = \frac{AZ}{BZ},$$

so $Z' = Z$. It follows that CZ' and CZ are the same segment, so AX , BY , CZ are concurrent.

Ceva's theorem has a number of immediate consequences.

1. If X , Y , Z are the midpoints of the sides of the triangle, then by Ceva's theorem the cevians are easily seen to be concurrent. In this case the cevians are usually called *medians*, so we know that the three medians of any triangle intersect at a single point.

2. Ceva's theorem also implies that the three altitudes from the vertices to the opposite sides are concurrent. The proof of this requires a little trigonometry (Fig. 53): if the sides of the triangle are denoted by a , b , c , then $AZ = b \cos A$, $BZ = a \cos B$, $BX = c \cos B$, $CX = b \cos C$, $CY = a \cos C$ and $AY = c \cos A$, so

$$\frac{AZ}{BZ} \cdot \frac{BX}{CX} \cdot \frac{CY}{AY} = \frac{b \cos A}{a \cos B} \cdot \frac{c \cos B}{b \cos C} \cdot \frac{a \cos C}{c \cos A} = 1.$$

3. In the 19th century a French mathematician named Joseph Gergonne proved the following: if a circle is inscribed in a triangle ABC (Fig. 54), and if X , Y , Z are the points where the circle is tangent to the sides of the triangle, then AX , BY and CZ are concurrent. Why is this true?

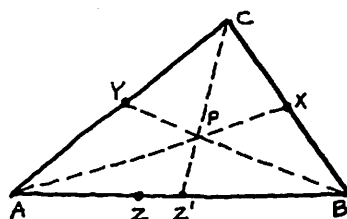


FIG. 52

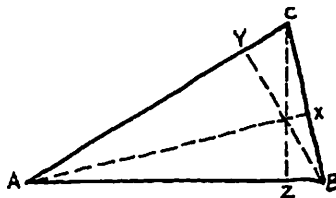


FIG. 53

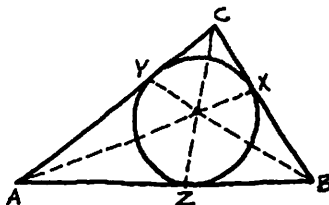


FIG. 54

APPENDIX F. BRAHMAGUPTA'S FORMULA

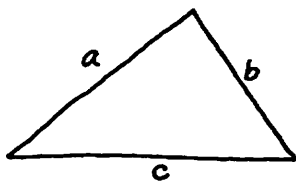


FIG. 55

In Exercise 20 of Section 1 we ask the reader to prove Heron's formula for the area A of a triangle with sides a, b, c (Fig. 55):

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a + b + c)$ is the semiperimeter of

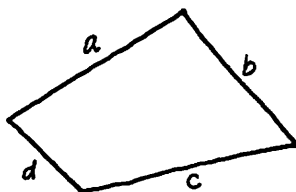


FIG. 56

the triangle. The presence of the factor s under the radical suggests that this formula might be a special case of a more general formula

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)} \quad (*)$$

giving the area of a quadrilateral with sides a, b, c, d , where $s = \frac{1}{2}(a + b + c + d)$ is the semi-

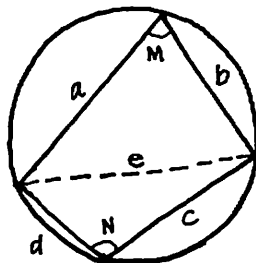


FIG. 57

perimeter of the quadrilateral (Fig. 56). After all, if the side d shrinks to zero, then the quadrilateral becomes a triangle and formula (*) collapses to Heron's formula, which we know is correct. Unfortunately this conjecture is false (can you see at once why it cannot be true?). However, a modified version is true: if the quadrilateral is inscribed in a circle (Fig. 57), then formula (*) is valid. Under these circumstances (*) is called Brahmagupta's formula, after the 7th century Indian mathematician who discovered it.

The proof we give makes use of trigonometry. We begin by inserting the diagonal e in the quadrilateral of Fig. 57, and also by labeling the opposite angles M and N . By Exercise 10 in Section 2, $M + N = 180^\circ$, so

$$\cos N = -\cos M \quad \text{and} \quad \sin N = \sin M.$$

By the law of cosines,

$$a^2 + b^2 - 2ab \cos M = e^2 = c^2 + d^2 - 2cd \cos N,$$

so

$$2(ab + cd)\cos M = a^2 + b^2 - c^2 - d^2. \quad (**)$$

Since the area A of the quadrilateral is given by

$$A = \frac{1}{2}ab \sin M + \frac{1}{2}cd \sin N = \frac{1}{2}(ab + cd)\sin M,$$

we also have

$$2(ab + cd)\sin M = 4A. \quad (***)$$

By squaring and adding equations (**) and (***),

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and using the identity $\sin^2 M + \cos^2 M = 1$, we obtain

$$4(ab + cd)^2 = (a^2 + b^2 - c^2 - d^2)^2 + 16A^2,$$

so

$$16A^2 = (2ab + 2cd)^2 - (a^2 + b^2 - c^2 - d^2)^2.$$

By repeatedly factoring differences of two squares in accordance with the identity $x^2 - y^2 = (x + y) \times (x - y)$, we obtain

$$\begin{aligned} 16A^2 &= [2ab + 2cd + a^2 + b^2 - c^2 - d^2] \\ &\quad \cdot [2ab + 2cd - a^2 - b^2 + c^2 + d^2] \\ &= [(a + b)^2 - (c - d)^2] \\ &\quad \times [(c + d)^2 - (a - b)^2] \\ &= [a + b + c - d][a + b - c + d] \\ &\quad \times [c + d + a - b][c + d - a + b] \\ &= [2s - 2d][2s - 2c][2s - 2b][2s - 2a] \end{aligned}$$

or

$$A^2 = (s - a)(s - b)(s - c)(s - d),$$

which proves Brahmagupta's formula.